

# GAP computations needed in the proof of [DNT, Theorem 6.1 (ii)]

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## Abstract

This is a collection of example computations that are cited in the Appendix of [DNT]. In each case, the aim is to show that the extension of a given finite simple group by an elementary abelian group of given rank has the property that not all complex irreducible characters of the same degree are Galois conjugate.

The purpose of this writeup is twofold. On the one hand, the details of the computations are documented this way. On the other hand, the GAP code shown for the examples can be used as test input for automatic checking of the data and the functions used.

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For the computations, we need some Brauer character tables from [JLPW95], some generating matrices from [WWT<sup>+</sup>], and some functions from the GAP system [GAP12] and its packages `AtlasRep` [WPN<sup>+</sup>11], `cohomolo` [Hol08], `CTb1Lib` [Bre12], and `TomLib` [NMP11].

First we load the necessary GAP packages.

```

gap> LoadPackage( "AtlasRep", "1.5" );
true
gap> LoadPackage( "cohomolo", "1.6" );
true
gap> LoadPackage( "CTblLib", "1.2" );
true
gap> LoadPackage( "TomLib", "1.2.1" );
true

```

## 1 $G/N \cong Sz(8)$ and $|N| = 2^{12}$

The group  $S = Sz(8)$  has exactly one irreducible 12-dimensional module over the field with two elements, up to isomorphism. This module can be obtained from any of the three absolutely irreducible 4-dimensional  $S$ -modules in characteristic two, by regarding it as a module over the prime field  $\mathbb{F}_2$ .

```

gap> p:= 2;; d:= 12;;
gap> t:= CharacterTable( "Sz(8)" ) mod p;
BrauerTable( "Sz(8)", 2 )
gap> irr:= Filtered( Irr( t ), x -> x[1] <= d );;
gap> Display( t, rec( chars:= irr, powermap:= false, centralizers:= false ) );
Sz(8)mod2

```

	1a	5a	7a	7b	7c	13a	13b	13c
Y.1	1	1	1	1	1	1	1	1
Y.2	4	-1	A	C	B	D	F	E
Y.3	4	-1	B	A	C	E	D	F
Y.4	4	-1	C	B	A	F	E	D

```

A = E(7)^2+E(7)^3+E(7)^4+E(7)^5
B = E(7)+E(7)^2+E(7)^5+E(7)^6
C = E(7)+E(7)^3+E(7)^4+E(7)^6
D = E(13)+E(13)^5+E(13)^8+E(13)^12
E = E(13)^4+E(13)^6+E(13)^7+E(13)^9
F = E(13)^2+E(13)^3+E(13)^10+E(13)^11
gap> List( irr, x -> SizeOfFieldOfDefinition( x, p ) );
[ 2, 8, 8, 8 ]

```

First we construct the 12-dimensional irreducible representation of  $S$  over  $\mathbb{F}_2$ , using that the ATLAS of Group Representations provides matrix generators for  $S$  in the 4-dimensional representation over  $\mathbb{F}_8$ .

```

gap> info:= OneAtlasGeneratingSetInfo( "Sz(8)", Dimension, 4,
> Characteristic, p );
rec( charactername := "4a", dim := 4, groupname := "Sz(8)", id := "a",
  identifier := [ "Sz(8)", [ "Sz8G1-f8r4aB0.m1", "Sz8G1-f8r4aB0.m2" ], 1, 8 ],
  repname := "Sz8G1-f8r4aB0", repnr := 17, ring := GF(2^3), size := 29120,
  standardization := 1, type := "matff" )
gap> gens_dim4:= AtlasGenerators( info ).generators;;
gap> b:= Basis( GF(8) );;
gap> gens_dim12:= List( gens_dim4, x -> BlownUpMatrix( b, x ) );;

```

We claim that any extension of  $S$  with the given module splits.

```

gap> s:= AtlasGroup( "Sz(8)", IsPermGroup, true );;
gap> chr:= CHR( s, p, 0, gens_dim12 );;
gap> SecondCohomologyDimension( chr );
0

```

(The function `CHR` takes as its arguments a permutation group, the characteristic of the module, a finitely presented group (or zero), and a list of matrices that define the module in the sense that they correspond to the generators of the given permutation group. Note that this condition is satisfied because the generators provided by the ATLAS of Group Representations are compatible.)

So it is enough to consider the semidirect product  $G = 2^{12}:Sz(8)$ .

The GAP Character Table Library contains the ordinary character table of  $G$ . We check this as follows. By the above cohomology result, the group  $G$  is uniquely determined, up to isomorphism, by the group order and the property that  $G$  has a minimal normal subgroup  $N$  such that  $G/N$  is a simple group isomorphic with  $S$ . (Since  $|G|/|S|$  is a power of two,  $N$  is a 2-group. By the minimality condition,  $N$  is elementary abelian and the action of  $S$  on  $N$  affords the desired  $S$ -module. Note that the isomorphism type of a finite simple group is determined by its character table.)

```

gap> iso:= IsomorphismTypeInfoFiniteSimpleGroup( s );
rec( name := "2B(2,8) = 2C(2,8) = Sz(8)", parameter := 8, series := "2B" )
gap> names:= AllCharacterTableNames( Size, 2^12 * Size( s ) );;
gap> cand:= List( names, CharacterTable );;
gap> cand:= Filtered( cand,
>   t -> ForAny( ClassPositionsOfMinimalNormalSubgroups( t ),
>     n -> IsomorphismTypeInfoFiniteSimpleGroup( t / n ) = iso );
[ CharacterTable( "2^12:Sz(8)" ) ]

```

So we can easily check that  $G$  has eight rational valued irreducibles of the degree 455 (or of the degree 3640).

```

gap> t:= cand[1];;
gap> rationals:= Filtered( Irr( t ), x -> IsSubset( Integers, x ) );;
gap> Collected( List( rationals, x -> x[1] ) );
[ [ 1, 1 ], [ 64, 1 ], [ 91, 1 ], [ 455, 8 ], [ 3640, 8 ] ]

```

## 2 $G/N \cong M_{22}$ and $|N| = 2^{10}$

The group  $S = M_{22}$  has exactly two irreducible 10-dimensional modules over the field with two elements, up to isomorphism. These modules are in fact absolutely irreducible.

```

gap> p:= 2;; d:= 10;;
gap> t:= CharacterTable( "M22" ) mod p;
BrauereTable( "M22", 2 )
gap> irr:= Filtered( Irr( t ), x -> x[1] <= d );;
gap> Display( t, rec( chars:= irr, powermap:= false, centralizers:= false ) );
M22mod2

```

```

      1a 3a 5a 7a 7b 11a 11b

```

```

Y.1    1  1  1  1  1  1  1
Y.2   10  1  .  A /A -1 -1
Y.3   10  1  . /A  A -1 -1

```

```

A = E(7)+E(7)^2+E(7)^4

```

```

    = (-1+sqrt(-7))/2 = b7
gap> List( irr, x -> SizeOfFieldOfDefinition( x, p ) );
[ 2, 2, 2 ]

```

First we construct the two irreducible 10-dimensional representations of  $S$  over  $\mathbb{F}_2$ , again using that the ATLAS of Group Representations provides the matrix generators in question.

```

gap> info:= AllAtlasGeneratingSetInfos( "M22", Dimension, d,
>
> Characteristic, p );
[ rec( charactername := "10a", dim := 10, groupname := "M22", id := "a",
  identifier := [ "M22", [ "M22G1-f2r10aB0.m1", "M22G1-f2r10aB0.m2" ], 1,
    2 ], repname := "M22G1-f2r10aB0", repr := 13, ring := GF(2),
  size := 443520, standardization := 1, type := "matff" ),
  rec( charactername := "10b", dim := 10, groupname := "M22", id := "b",
  identifier := [ "M22", [ "M22G1-f2r10bB0.m1", "M22G1-f2r10bB0.m2" ], 1,
    2 ], repname := "M22G1-f2r10bB0", repr := 14, ring := GF(2),
  size := 443520, standardization := 1, type := "matff" ) ]
gap> gens:= List( info, r -> AtlasGenerators( r ).generators );;

```

We claim that any extension of  $S$  with any of the two given modules splits.

```

gap> s:= AtlasGroup( "M22", IsPermGroup, true );;
gap> chr:= CHR( s, p, 0, gens[1] );;
gap> SecondCohomologyDimension( chr );
0
gap> chr:= CHR( s, p, 0, gens[2] );;
gap> SecondCohomologyDimension( chr );
0

```

Again we see that it is enough to consider semidirect products  $G = 2^{10} : M_{22}$ , but this time for the two nonisomorphic modules.

The GAP Character Table Library contains the ordinary character tables of the two groups in question. We check this with the same approach as in the previous examples.

```

gap> iso:= IsomorphismTypeInfoFiniteSimpleGroup( s );
rec( name := "M(22)", series := "Spor" )
gap> names:= AllCharacterTableNames( Size, 2^10 * Size( s ) );;
gap> cand:= List( names, CharacterTable );;
gap> cand:= Filtered( cand,
>
> t -> ForAny( ClassPositionsOfMinimalNormalSubgroups( t ),
>
> n -> IsomorphismTypeInfoFiniteSimpleGroup( t / n ) = iso );
[ CharacterTable( "2^10:M22'" ), CharacterTable( "2^10:m22" ) ]
gap> List( cand, NrConjugacyClasses );
[ 47, 43 ]

```

So we can easily check that in both cases,  $G$  has two rational valued irreducibles of the degree 1155.

```

gap> t:= cand[1];;
gap> rationals:= Filtered( Irr( t ), x -> IsSubset( Integers, x ) );;
gap> Collected( List( rationals, x -> x[1] ) );
[ [ 1, 1 ], [ 21, 1 ], [ 22, 1 ], [ 55, 1 ], [ 99, 1 ], [ 154, 1 ],
  [ 210, 1 ], [ 231, 3 ], [ 385, 1 ], [ 440, 1 ], [ 770, 5 ], [ 924, 2 ],
  [ 1155, 2 ], [ 1386, 1 ], [ 1408, 1 ], [ 3080, 2 ], [ 3465, 4 ],
  [ 4620, 2 ], [ 6930, 3 ], [ 9240, 1 ] ]
gap> t:= cand[2];;

```

```

gap> rationals:= Filtered( Irr( t ), x -> IsSubset( Integers, x ) );
gap> Collected( List( rationals, x -> x[1] ) );
[ [ 1, 1 ], [ 21, 1 ], [ 55, 1 ], [ 77, 1 ], [ 99, 1 ], [ 154, 1 ],
  [ 210, 1 ], [ 231, 1 ], [ 330, 1 ], [ 385, 3 ], [ 616, 2 ], [ 693, 1 ],
  [ 770, 1 ], [ 1155, 2 ], [ 1980, 1 ], [ 2310, 4 ], [ 2640, 1 ],
  [ 3465, 2 ], [ 4620, 1 ], [ 5544, 2 ], [ 6160, 1 ], [ 6930, 2 ],
  [ 9856, 1 ] ]

```

### 3 $G/N \cong J_2$ and $|N| = 2^{12}$

The group  $S = J_2$  has exactly one irreducible 12-dimensional module over the field with two elements, up to isomorphism. This module can be obtained from any of the two absolutely irreducible 6-dimensional  $S$ -modules in characteristic two, by regarding it as a module over the prime field  $\mathbb{F}_2$ .

```

gap> p:= 2;; d:= 12;;
gap> t:= CharacterTable( "J2" ) mod p;
BrauerTable( "J2", 2 )
gap> irr:= Filtered( Irr( t ), x -> x[1] <= d );
gap> Display( t, rec( chars:= irr, powermap:= false, centralizers:= false ) );
J2mod2

      1a 3a 3b 5a 5b 5c 5d 7a 15a 15b

Y.1   1  1  1  1  1  1  1  1  1  1
Y.2   6 -3  .  A *A B *B -1  C *C
Y.3   6 -3  . *A  A *B B -1  *C  C

A = -2*E(5)-2*E(5)^4
    = 1-Sqrt(5) = 1-r5
B = E(5)+2*E(5)^2+2*E(5)^3+E(5)^4
    = (-3-Sqrt(5))/2 = -2-b5
C = E(5)+E(5)^4
    = (-1+Sqrt(5))/2 = b5
gap> List( irr, x -> SizeOfFieldOfDefinition( x, p ) );
[ 2, 4, 4 ]

```

First we construct the irreducible 12-dimensional representation of  $S$  over  $\mathbb{F}_2$ , using that the ATLAS of Group Representations provides matrix generators for  $S$  in the 6-dimensional representation over  $\mathbb{F}_4$ .

```

gap> info:= OneAtlasGeneratingSetInfo( "J2", Dimension, 6,
> Characteristic, p );
rec( charactername := "6a", dim := 6, groupname := "J2", id := "a",
  identifier := [ "J2", [ "J2G1-f4r6aB0.m1", "J2G1-f4r6aB0.m2" ], 1, 4 ],
  repname := "J2G1-f4r6aB0", repnr := 16, ring := GF(2^2), size := 604800,
  standardization := 1, type := "matff" )
gap> gens_dim6:= AtlasGenerators( info ).generators;;
gap> b:= Basis( GF(4) );
gap> gens_dim12:= List( gens_dim6, x -> BlownUpMatrix( b, x ) );

```

We claim that any extension of  $S$  with the given module splits.

```

gap> s:= AtlasGroup( "J2", IsPermGroup, true );
gap> chr:= CHR( s, p, 0, gens_dim12 );

```

```
gap> SecondCohomologyDimension( chr );
0
```

Again we see that it is enough to consider a semidirect product  $G = 2^{12}:J_2$ .

The GAP Character Table Library contains the ordinary character table of  $G$ . We check this with the same approach as in the previous examples.

```
gap> iso:= IsomorphismTypeInfoFiniteSimpleGroup( s );
rec( name := "HJ = J(2) = F(5-)", series := "Spor" )
gap> names:= AllCharacterTableNames( Size, 2^12 * Size( s ) );
gap> cand:= List( names, CharacterTable );
gap> cand:= Filtered( cand,
>   t -> ForAny( ClassPositionsOfMinimalNormalSubgroups( t ),
>     n -> IsomorphismTypeInfoFiniteSimpleGroup( t / n ) = iso ) );
[ CharacterTable( "2^12:J2" ) ]
```

So we can easily check that  $G$  has two rational valued irreducibles of the degree 1575.

```
gap> t:= cand[1];
gap> rationals:= Filtered( Irr( t ), x -> IsSubset( Integers, x ) );
gap> Collected( List( rationals, x -> x[1] ) );
[ [ 1, 1 ], [ 36, 1 ], [ 63, 1 ], [ 90, 1 ], [ 126, 1 ], [ 160, 1 ],
  [ 175, 1 ], [ 225, 1 ], [ 288, 1 ], [ 300, 1 ], [ 336, 1 ], [ 1575, 2 ],
  [ 2520, 4 ], [ 3150, 1 ], [ 4725, 6 ], [ 9450, 1 ], [ 10080, 4 ],
  [ 12600, 4 ], [ 18900, 2 ] ]
```

## 4 $G/N \cong J_2$ and $|N| = 5^{14}$

The group  $S = J_2$  has exactly one irreducible 14-dimensional module over the field with 5 elements, up to isomorphism. This module is in fact absolutely irreducible.

```
gap> p:= 5;; d:= 14;;
gap> t:= CharacterTable( "J2" ) mod p;
BrauerTable( "J2", 5 )
gap> irr:= Filtered( Irr( t ), x -> x[1] <= d );
gap> Display( t, rec( chars:= irr, powermap:= false, centralizers:= false ) );
J2mod5
```

```
1a 2a 2b 3a 3b 4a 6a 6b 7a 8a 12a
```

```
Y.1   1  1  1  1  1  1  1  1  1  1  1
Y.2  14 -2  2  5 -1  2  1 -1  .  . -1
```

In this case, we do not attempt to compute the complete character table of  $G$ . Instead, we show that  $G/N$  has at least five regular orbits on the dual space of  $N$ , and apply [DNT, Lemma 5.1 (i)]. (Note that  $N$  is in fact self-dual.)

For that, we use GAP's table of marks of  $S$ . The information stored for this table of marks allows us to compute, for each class of subgroups  $U$  of  $S$ , the numbers of orbits in the dual space of  $N$  for which contain the point stabilizers in  $S$  are exactly the conjugates of  $U$ . The following GAP function takes the table of marks `tom` of  $S$ , a list `matgens` of matrices that describe the action of the generators of  $S$  on the vector space in question, and the size `q` of its field of scalars. The return value is a record with the components `fixed` (the vector of numbers of fixed points of the subgroups of  $S$  on the dual of  $N$ ), `decomp` (the numbers of orbits with the corresponding point stabilizers), `nonzeropos` (the positions of subgroups that occur as point stabilizers), and `staborders` (the list of orders of the subgroups that occur as point stabilizers).

```

gap> orbits_from_tom:= function( tom, matgens, q )
>   local slp, fixed, idmat, i, rest, decomp, nonzeropos;
>
>   slp:= StraightLineProgramsTom( tom );
>   fixed:= [];
>   idmat:= matgens[1]^0;
>   for i in [ 1 .. Length( slp ) ] do
>     if IsList( slp[i] ) then
>       # Each subgroup generator has a program of its own.
>       rest:= List( slp[i],
>                   prg -> ResultOfStraightLineProgram( prg, gens ) );
>     else
>       # The subgroup generators are computed with one common program.
>       rest:= ResultOfStraightLineProgram( slp[i], gens );
>       fi;
>       if IsEmpty( rest ) then
>         # The subgroup is trivial.
>         fixed[i]:= q^Length( idmat );
>       else
>         # Compute the intersection of fixed spaces of the transposed
>         # matrices, since we act on Irr(N) not on N.
>         fixed[i]:= q^Length( NullspaceMat( TransposedMat( Concatenation(
>           List( rest, x -> x - idmat ) ) ) ) );
>       fi;
>     od;
>
>   decomp:= DecomposedFixedPointVector( tom, fixed );
>   nonzeropos:= Filtered( [ 1 .. Length( decomp ) ],
>                          i -> decomp[i] <> 0 );
>
>   return rec( fixed:= fixed,
>              decomp:= decomp,
>              nonzeropos:= nonzeropos,
>              staborders:= OrdersTom( tom ){ nonzeropos },
>              );
> end;;

```

Note that this function assumes that the generators of  $S$  obtained from the ATLAS of Group Representations are compatible with the generators from GAP's table of marks of  $S$ . This fact can be read off from the `true` value of the ATLAS component in the `StandardGeneratorsInfo` value of the table of marks.

```

gap> tom:= TableOfMarks( "J2" );
TableOfMarks( "J2" )
gap> StandardGeneratorsInfo( tom );
[ rec( ATLAS := true, description := "|z|=10, z^5=a, |b|=3, |C(b)|=36, |ab|=7"
, generators := "a, b",
script := [ [ 1, 10, 5 ], [ 2, 3 ], [ [ 2, 1 ], [ "|C(", , ")|" ], 36 ],
[ 1, 1, 2, 1, 7 ] ], standardization := 1 ) ]

```

Alternatively, we can compute whether the generators are compatible, as follows.

```

gap> info:= OneAtlasGeneratingSetInfo( "J2", Dimension, d, Ring, GF(p) );
rec( charactername := "14a", dim := 14, groupname := "J2", id := "",
identifier := [ "J2", [ "J2G1-f5r14B0.m1", "J2G1-f5r14B0.m2" ], 1, 5 ],

```

```

    repname := "J2G1-f5r14B0", repnr := 19, ring := GF(5), size := 604800,
    standardization := 1, type := "matff" )
gap> gens:= AtlasGenerators( info ).generators;;
gap> map:= GroupGeneralMappingByImages( UnderlyingGroup( tom ),
>    Group( gens ), GeneratorsOfGroup( UnderlyingGroup( tom ) ), gens );;
gap> IsGroupHomomorphism( map );
true

```

Now we are sure that we may apply the function `orbits_from_tom`.

```

gap> orbits_from_tom( tom, gens, p );
rec( decomp := [ 8600, 0, 2512, 359, 10, 0, 0, 212, 5, 0, 0, 4, 0, 240, 16,
    10, 0, 0, 0, 0, 10, 0, 0, 0, 0, 2, 0, 0, 36, 0, 0, 0, 26, 0, 0, 0, 0,
    0, 0, 0, 20, 0, 10, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 10, 0, 0,
    5, 0, 0, 0, 26, 0, 10, 0, 0, 0, 0, 10, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 10, 0, 0, 0, 2, 0, 0, 0, 0, 2, 4, 0, 0, 0, 0, 0, 4, 0,
    0, 0, 0, 0, 0, 0, 0, 0, 16, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 4, 0, 0, 1 ],
    fixed := [ 6103515625, 15625, 390625, 390625, 625, 25, 3125, 3125, 625,
    625, 625, 625, 5, 3125, 125, 625, 25, 25, 125, 5, 125, 25, 125, 25, 25,
    25, 5, 125, 125, 125, 25, 25, 3125, 1, 1, 5, 5, 25, 5, 25, 125, 5, 25,
    25, 25, 25, 25, 25, 5, 25, 25, 5, 25, 5, 5, 5, 5, 25, 25, 1, 125, 1, 5,
    5, 125, 1, 25, 5, 25, 1, 5, 25, 5, 5, 25, 25, 5, 5, 5, 1, 5, 5, 1, 1,
    1, 5, 1, 25, 25, 25, 1, 5, 25, 5, 5, 1, 1, 125, 5, 5, 5, 25, 5, 5, 5,
    1, 1, 5, 5, 1, 5, 1, 5, 1, 1, 25, 5, 5, 1, 1, 1, 1, 5, 1, 1, 25, 1, 1,
    5, 1, 1, 5, 1, 5, 1, 1, 5, 1, 5, 1, 1, 1, 5, 1, 1, 1 ],
    nonzeropos := [ 1, 3, 4, 5, 8, 9, 12, 14, 15, 16, 21, 26, 29, 33, 41, 43,
    44, 58, 61, 65, 67, 72, 89, 93, 98, 99, 105, 116, 126, 139, 143, 146 ],
    staborders := [ 1, 2, 3, 3, 4, 4, 5, 6, 6, 6, 8, 9, 10, 12, 12, 12, 14, 20,
    24, 24, 24, 30, 48, 50, 60, 60, 72, 120, 192, 600, 1920, 604800 ] )

```

We see that  $S$  has 8600 regular orbits on (the dual space of)  $N$ .

## 5 $G/N \cong J_2$ and $|N| = 2^{28}$

The group  $S = J_2$  has exactly one irreducible 28-dimensional module over the field with two elements, up to isomorphism. This module can be obtained from any of the two absolutely irreducible 14-dimensional  $S$ -modules in characteristic two, by regarding it as a module over the prime field  $\mathbb{F}_2$ .

```

gap> p:= 2;; d:= 28;;
gap> t:= CharacterTable( "J2" ) mod p;
BrauerTable( "J2", 2 )
gap> irr:= Filtered( Irr( t ), x -> x[1] <= d );;
gap> Display( t, rec( chars:= irr, powermap:= false, centralizers:= false ) );
J2mod2

```

1a 3a 3b 5a 5b 5c 5d 7a 15a 15b

```

Y.1  1  1  1  1  1  1  1  1  1  1
Y.2  6 -3  .  A *A  C  *C -1  D  *D
Y.3  6 -3  . *A  A  *C  C -1  *D  D
Y.4  14 5 -1  B *B  -C -*C  .  .  .
Y.5  14 5 -1 *B  B  -*C -C  .  .  .

```

```

A = -2*E(5)-2*E(5)^4
  = 1-Sqrt(5) = 1-r5
B = -3*E(5)-3*E(5)^4
  = (3-3*Sqrt(5))/2 = -3b5
C = E(5)+2*E(5)^2+2*E(5)^3+E(5)^4
  = (-3-Sqrt(5))/2 = -2-b5
D = E(5)+E(5)^4
  = (-1+Sqrt(5))/2 = b5
gap> List( irr, x -> SizeOfFieldOfDefinition( x, p ) );
[ 2, 4, 4, 4, 4 ]

```

We use the same approach as in the previous example.

```

gap> tom:= TableOfMarks( "J2" );;
gap> info:= OneAtlasGeneratingSetInfo( "J2", Dimension, 14, Ring, GF(4) );;
gap> gens:= List( AtlasGenerators( info ).generators,
>               x -> BlownUpMat( Basis(GF(4)), x ) );;
gap> orbits_from_tom( tom, gens, p );
rec( decomp := [ 235, 33, 282, 38, 0, 0, 6, 31, 36, 0, 0, 0, 3, 66, 9, 0, 0,
0, 0, 0, 0, 2, 18, 0, 0, 1, 0, 0, 15, 0, 0, 0, 6, 0, 0, 0, 0, 0, 0, 0,
12, 0, 0, 5, 0, 1, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 1, 3,
0, 9, 0, 0, 0, 0, 0, 0, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 3, 0, 0, 0, 6, 0, 0,
0, 0, 0, 0, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0,
0, 0, 0, 3, 0, 0, 0, 3, 0, 0, 1 ],
fixed := [ 268435456, 65536, 65536, 65536, 256, 1024, 4096, 1024, 1024,
256, 256, 256, 64, 1024, 64, 256, 16, 16, 64, 64, 64, 256, 256, 64, 16,
16, 64, 64, 64, 64, 16, 16, 1024, 4, 4, 4, 4, 16, 16, 16, 64, 16, 16,
16, 16, 64, 16, 16, 16, 64, 16, 16, 16, 16, 4, 16, 16, 16, 16, 1, 64,
4, 16, 4, 64, 4, 16, 4, 16, 1, 4, 16, 4, 4, 16, 16, 4, 4, 16, 1, 4, 16,
1, 1, 1, 16, 4, 16, 16, 16, 1, 4, 16, 4, 4, 1, 4, 64, 4, 4, 4, 16, 4,
4, 4, 1, 1, 4, 16, 1, 4, 1, 4, 1, 4, 16, 4, 4, 1, 1, 1, 1, 4, 1, 1, 16,
1, 1, 4, 1, 4, 4, 1, 4, 1, 1, 4, 1, 4, 1, 1, 1, 4, 1, 1, 1 ],
nonzeropos := [ 1, 2, 3, 4, 7, 8, 9, 13, 14, 15, 22, 23, 26, 29, 33, 41,
44, 46, 50, 61, 62, 63, 65, 72, 82, 93, 99, 105, 109, 116, 126, 131,
139, 143, 146 ],
staborders := [ 1, 2, 2, 3, 4, 4, 4, 6, 6, 6, 8, 8, 9, 10, 12, 12, 14, 16,
16, 24, 24, 24, 24, 30, 40, 50, 60, 72, 96, 120, 192, 240, 600, 1920,
604800 ] )

```

We see that  $S$  has 235 regular orbits on (the dual space of)  $N$ .

## 6 $G/N \cong {}^3D_4(2)$ and $|N| = 2^{26}$

The group  $S = {}^3D_4(2)$  has exactly one irreducible 26-dimensional module over the field with two elements, up to isomorphism. This module is in fact absolutely irreducible.

```

gap> p:= 2;; d:= 26;;
gap> t:= CharacterTable( "3D4(2)" ) mod p;
BrauerTable( "3D4(2)", 2 )
gap> irr:= Filtered( Irr( t ), x -> x[1] <= d );;
gap> Display( t, rec( chars:= irr, powermap:= false, centralizers:= false ) );
3D4(2)mod2

```

	1a	3a	3b	7a	7b	7c	7d	9a	9b	9c	13a	13b	13c	21a	21b	21c
Y.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Y.2	8	2	-1	A	C	B	1	D	F	E	G	I	H	J	L	K
Y.3	8	2	-1	B	A	C	1	E	D	F	H	G	I	K	J	L
Y.4	8	2	-1	C	B	A	1	F	E	D	I	H	G	L	K	J
Y.5	26	-1	-1	5	5	5	-2	2	2	2	.	.	.	-1	-1	-1

$A = 3 \cdot E(7)^2 + E(7)^3 + E(7)^4 + 3 \cdot E(7)^5$   
 $B = 3 \cdot E(7) + E(7)^2 + E(7)^5 + 3 \cdot E(7)^6$   
 $C = E(7) + 3 \cdot E(7)^3 + 3 \cdot E(7)^4 + E(7)^6$   
 $D = -E(9)^2 + E(9)^3 - 2 \cdot E(9)^4 - 2 \cdot E(9)^5 + E(9)^6 - E(9)^7$   
 $E = -E(9)^2 + E(9)^3 + E(9)^4 + E(9)^5 + E(9)^6 - E(9)^7$   
 $F = 2 \cdot E(9)^2 + E(9)^3 + E(9)^4 + E(9)^5 + E(9)^6 + 2 \cdot E(9)^7$   
 $G = E(13) + E(13)^2 + E(13)^3 + E(13)^5 + E(13)^8 + E(13)^{10} + E(13)^{11} + E(13)^{12}$   
 $H = E(13) + E(13)^4 + E(13)^5 + E(13)^6 + E(13)^7 + E(13)^8 + E(13)^9 + E(13)^{12}$   
 $I = E(13)^2 + E(13)^3 + E(13)^4 + E(13)^6 + E(13)^7 + E(13)^9 + E(13)^{10} + E(13)^{11}$   
 $J = E(7)^3 + E(7)^4$   
 $K = E(7)^2 + E(7)^5$   
 $L = E(7) + E(7)^6$

We try the same approach as in the examples about the group  $J_2$ .

```

gap> tom:= TableOfMarks( "3D4(2)" );
TableOfMarks( "3D4(2)" )
gap> StandardGeneratorsInfo( tom );
[ rec( ATLAS := true, description := "|z|=8, z^4=a, |b|=9, |ab|=13, |abb|=8",
      generators := "a, b",
      script := [ [ 1, 8, 4 ], [ 2, 9 ], [ 1, 1, 2, 1, 13 ],
                  [ 1, 1, 2, 1, 2, 1, 8 ] ], standardization := 1 ) ]
gap> info:= OneAtlasGeneratingSetInfo( "3D4(2)", Dimension, 26, Ring, GF(2) );;
gap> gens:= AtlasGenerators( info ).generators;;
gap> map:= GroupGeneralMappingByImages( UnderlyingGroup( tom ),
> Group( gens ), GeneratorsOfGroup( UnderlyingGroup( tom ) ), gens );;
gap> IsGroupHomomorphism( map );
true

```

Now we apply the function `orbits_from_tom`.

```

gap> orbsinfo:= orbits_from_tom( tom, gens, p );;
gap> orbsinfo.fixed[1];
67108864
gap> orbsinfo.decomp[1];
0

```

Unfortunately,  $S$  has no regular orbit on (the dual of)  $N$ . However, there is one orbit whose point stabilizer in  $S$  is a dihedral group  $D_{18}$  of order 18.

```

gap> orbsinfo.staborders;
[ 16, 16, 18, 42, 48, 52, 64, 72, 392, 1008, 1536, 3024, 3072, 3584, 258048,
  211341312 ]
gap> orbsinfo.nonzeropos[3];
446
gap> orbsinfo.decomp[446];

```

```

1
gap> u:= RepresentativeTom( tom, 446 );
<permutation group of size 18 with 2 generators>
gap> IsDihedralGroup( u );
true

```

Thus there is a linear character  $\lambda$  of  $N$  whose inertia subgroup  $T = I_G(\lambda)$  has the structure  $N.D_{18}$ . Now  $\text{Irr}(T|\lambda)$  can be identified with those irreducibles of  $T/\ker(\lambda)$  that restrict nontrivially to  $N/\ker(\lambda)$ , and there are only two groups, up to isomorphism, that can occur as  $T/\ker(\lambda)$ .

```

gap> cand:= Filtered( AllSmallGroups( 36 ),
>                   x -> Size( Centre( x ) ) = 2 and
>                   IsDihedralGroup( x / Centre( x ) ) );
[ <pc group of size 36 with 4 generators>,
  <pc group of size 36 with 4 generators> ]
gap> List( cand, StructureDescription );
[ "C9 : C4", "D36" ]

```

These two groups are a split and a nonsplit extension of the cyclic group of order 18 with a group of order two that acts by inverting. In other words, these two groups are the direct product of  $D_{18}$  with a cyclic group of order two and the subdirect product of  $D_{18}$  with a cyclic group of order four. Both groups possess irreducible characters of degree two, one rational valued and the other not, which restrict nontrivially to the centre.

```

gap> Display( CharacterTable( "Dihedral", 18 ) );
Dihedral(18)

```

```

      2  1  .  .  .  .  1
      3  2  2  2  2  2  .

      1a 9a 9b 3a 9c 2a
2P 1a 9b 9c 3a 9a 1a
3P 1a 3a 3a 1a 3a 2a

X.1   1  1  1  1  1  1
X.2   1  1  1  1  1 -1
X.3   2  A  B -1  C  .
X.4   2  B  C -1  A  .
X.5   2 -1 -1  2 -1  .
X.6   2  C  A -1  B  .

A = -E(9)^2-E(9)^4-E(9)^5-E(9)^7
B = E(9)^2+E(9)^7
C = E(9)^4+E(9)^5

```

By [DNT, Lemma 5.1 (ii)], we are done.

## 7 $G/N \cong {}^3D_4(2)$ and $|N| = 3^{25}$

The group  $S = {}^3D_4(2)$  has exactly one irreducible 25-dimensional module over the field with three elements, up to isomorphism. This module is in fact absolutely irreducible.

```

gap> p:= 3;; d:= 25;;
gap> t:= CharacterTable( "3D4(2)" ) mod p;
BrauerTable( "3D4(2)", 3 )
gap> irr:= Filtered( Irr( t ), x -> x[1] <= d );
gap> Display( t, rec( chars:= irr, powermap:= false, centralizers:= false ) );
3D4(2)mod3

```

	1a	2a	2b	4a	4b	4c	7a	7b	7c	7d	8a	8b	13a	13b	13c	14a	14b	14c	28a	28b	28c	
Y.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Y.2	25	-7	1	5	-3	1	4	4	4	-3	-1	-1	-1	-1	-1	.	.	.	-2	-2	-2	

We use the same approach as in the examples about the group  $J_2$ .

```

gap> tom:= TableOfMarks( "3D4(2)" );
gap> info:= OneAtlasGeneratingSetInfo( "3D4(2)", Dimension, d, Ring, GF(p) );
gap> gens:= AtlasGenerators( info ).generators;;
gap> orbsinfo:= orbits_from_tom( tom, gens, p );
gap> orbsinfo.fixed[1];
847288609443
gap> orbsinfo.decomp[1];
3551

```

We see that  $S$  has 3551 regular orbits on (the dual space of)  $N$ .

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